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Quantum Noether identities for non-local transformations in higher-order derivatives theories

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Abstract. Based on the phase-space generating functional of the Green function for a system with a regular/singular higher-order Lagrangian, the quantum canonical Noether identities (NIs) under a local and non-local transformation in phase space have been deduced, respectively. For a singular higher-order Lagrangian, one must use an effective canonical action $I^{\mathrm{P}}_{\mathrm{eff}}$ in quantum canonical NIs instead of the classical I^{P} in classical canonical NIs. The quantum NIs under a local and non-local transformation in configuration space for a gauge-invariant system with a higher-order Lagrangian have also been derived. The above results hold true whether or not the Jacobian of the transformation is equal to unity or not. It has been pointed out that in certain cases the quantum NIs may be converted to conservation laws at the quantum level. This algorithm to derive the quantum conservation laws is significantly different from the quantum first Noether theorem. The applications of our formulation to the Yang–Mills fields and non-Abelian Chern–Simons (CS) theories with higher-order derivatives are given, and the conserved quantities at the quantum level for local and non-local transformations are found, respectively.

1 Introduction

Local invariance is now a fundamental concept in modern field theories. Classical Noether identities (NIs) refer to the invariance of the action under a local transformation [1]; they play an important role in field theories [2]. Classical NIs and their generalization usually are formulated in terms of Lagrangian variables in configuration space. Classical NIs in the canonical formalism have been developed in previous works [3–5]. These canonical NIs are useful tools for the study of constraints of the system in Dirac's sense [3–5]. On application to Yang–Mills theories with higher-order derivatives, classical NIs may be converted into conservation laws along the trajectory of the motion [5,6]; in that case for deriving those conservation laws, the effective Lagrangian is obtained by using the Faddeev–Popov (FP) trick. Thus, this formulation is a semi-classical theory which is not constructed by making a thorough investigation in a totally quantum theory. This problem will be further discussed here at the quantum level. The quantum first Noether theorem had been established in previous works [7–9], and quantum NIs for a local transformation in first-derivatives theories have also been formulated [10]. Dynamical systems described in terms of a higher-order Lagrangian obtained by many authors are of much interest in connection with gauge theories, gravity, supersymmetry, string models, and other problems [4, 11]. In Yang–Mills theories and in using conformal symmetry of the gauge field theories and other problems, some non-local transformations were introduced [12–14]. Now the quantum NIs under the local and non-local transformation for a system with a higher-order Lagrangian will be further studied.

In the study of the symmetries at the quantum level, the path integral provides a useful tool. The phase-space path integrals are more basic than the configuration-space ones [15]. However, for a gauge-invariant system one can conveniently use the FP trick to formulate its path-integral quantization in configuration space. In certain integrable cases, according to the path-integral quantization of the constrained Hamiltonian system (for example, a gauge-invariant system), one can carry out explicit integration of the canonical momenta in the phase-space path integral, which may be converted to the same results obtained by using the FP trick (for example, Yang–Mills theories).

This paper is organized as follows. In Sect. 2, based on the phase-space generating functional of the Green function for a system with a higher-order Lagrangian, the quantum canonical NIs under a local and non-local transformation have been deduced. The form of these identities coincides with the classical ones for the regular higher-order Lagrangian, but for the singular higher-order La-

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grangian one must use an effective canonical action $I^{\mathrm{P}}_{\mathrm{eff}}$ instead of a canonical action I^{P} as in those quantum NIs. It is pointed out that these quantum NIs hold true no matter whether the Jacobian of the transformation is equal to unity or not. In Sect. 3, for a gauge-invariant system with a higher-order Lagrangian, based on a configuration-space generating functional obtained by using the FP trick, the quantum NIs under the local and non-local transformation in configuration space have been also derived. The expressions of these quantum NIs differ from classical ones in that one must use an effective action instead of the classical action in the corresponding terms. These configuration-space quantum NIs also hold true whether the Jacobian of the transformation is equal to unity or not. In Sect. 4, it is pointed out that, based on the quantum NIs, in certain cases one can find the quantum conservation laws of the system with a higher-order Lagrangian. This algorithm to derive the quantum conservation laws in quantum theory is significantly different from the quantum first Noether theorem [8,9]. In Sect. 5, we give a preliminary application of the above formulation to Yang–Mills fields with higher-order derivatives, and a new quantum conserved quantity is obtained. The application to the non-Abelian Chern–Simons theory with higher-order derivatives is given in Sect. 6; the quantum conserved BRS and PBRS quantities and other quantum conserved quantities connected with the non-local transformation are found. Section 7 is devoted to our conclusions and to a discussion.

2 Quantum canonical Noether identities for a non-local transformation

Let us first consider a physical field defined by n field variables $\varphi^\alpha(x)$ $(\alpha=1,2,\cdots,n)$, $x=(x_0,x_i)$ $(x_0=t, i=1,2,3)$, where the motion of the field is described by a regular Lagrangian involving higher-order derivatives of field variables, and the Lagrangian of the system is given by

$$L[\varphi^\alpha_{(0)},\varphi^\alpha_{(1)},\cdots,\varphi^\alpha_{(N)}]=\int_V \mathcal{L}(\varphi^\alpha_{,},\varphi^\alpha_{,\mu},\cdots,\varphi^\alpha_{,\mu(N)})\mathrm{d}^3x, \tag{2.1}$$

where $\varphi^\alpha_{(0)}=\varphi^\alpha, \varphi^\alpha_{(1)}=\dot{\varphi}^\alpha,\cdots,\varphi^\alpha_{,\mu}=\partial_\mu\varphi^\alpha, \varphi^\alpha_{,\mu(m)}=\underbrace{\partial_\mu\partial_\nu\cdots\partial_\rho}_{m}\varphi^\alpha$, and V is the space domain of the field. The flat space-time metric is $g_{\mu\nu}=(1-1-1-1)$. Using the Ostrogradsky transformation [16], one can introduce generalized canonical momenta:

$$\pi^{(N-1)}_\alpha=\frac{\delta L}{\delta\varphi^\alpha_{(N)}}, \tag{2.2}$$

$$\pi^{(s-1)}_\alpha=\frac{\delta L}{\delta\varphi^\alpha_{(s)}}-\dot{\pi}^{(s)}_\alpha \quad (s=1,2,\cdots,N-1), \tag{2.3}$$

or

$$\pi^{(s-1)}_\alpha=\sum_{j=0}^{N-s}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\delta L}{\delta\varphi^\alpha_{(j+s)}} \quad (s=1,2,\cdots,N), \tag{2.4}$$

and using these relations one can go over from the Lagrangian description to the Hamiltonian description. The generalized canonical Hamiltonian is defined by

$$H_{\mathrm{c}}[\varphi^\alpha_{(s)},\pi^{(s)}_\alpha]=\int\mathcal{H}_{\mathrm{c}}\mathrm{d}^3x=\int(\pi^{(s)}_\alpha\varphi^\alpha_{(s+1)}-\mathcal{L})\mathrm{d}^3x \quad (\varphi^\alpha_{(s)}=\dot{\varphi}^\alpha_{(s)}), \tag{2.5}$$

which may be formed by eliminating only the highest derivatives $\varphi^\alpha_{(N)}$. The summation over the indices α from 1 to n and s from 0 to $N-1$ are taken repeatedly.

The phase-space generating functional of the Green function for a system with a regular higher-order Lagrangian is given by [17]

$$Z[J,K]=\int\mathcal{D}\varphi^\alpha_{(s)}\mathcal{D}\pi^{(s)}_\alpha\exp\left\{\mathrm{i}\left[I^{\mathrm{P}}+\int\mathrm{d}^4x(J^s_\alpha\varphi^\alpha_{(s)}+K^\alpha_s\pi^{(s)}_\alpha)\right]\right\}, \tag{2.6}$$

where I^{P} is a canonical action,

$$I^{\mathrm{P}}=\int\mathrm{d}^4\mathcal{L}^{\mathrm{P}}=\int\mathrm{d}^4x(\pi^{(s)}_\alpha\varphi^\alpha_{(s+1)}-\mathcal{H}_{\mathrm{c}}). \tag{2.7}$$

J^s_α and K^α_s are exterior sources with respect to $\varphi^\alpha_{(s)}$ and $\pi^{(s)}_\alpha$, respectively.

Let us consider an infinitesimal local and non-local transformation in extended phase space

$$\begin{cases} x^{\mu'}=x^\mu+\Delta x^\mu=x^\mu+R^\mu_\sigma\varepsilon^\sigma(x),\\ \varphi^{\alpha'}_{(s)}(x')=\varphi^\alpha_{(s)}(x)+\Delta\varphi^\alpha_{(s)}(x)\\ \quad=\varphi^\alpha_{(s)}(x)+S^\alpha_{s\sigma}\varepsilon^\sigma(x)+\int\mathrm{d}^4yE(x,y)A^\alpha_{s\sigma}(y)\varepsilon^\sigma(y),\\ \pi^{(s)'}_\alpha(x')=\pi^{(s)}_\alpha(x)+\Delta\pi^{(s)}_\alpha(x)\\ \quad=\pi^{(s)}_\alpha(x)+T^s_{\alpha\sigma}\varepsilon^\sigma(x)+\int\mathrm{d}^4yF(x,y)B^s_{\alpha\sigma}(y)\varepsilon^\sigma(y), \end{cases} \tag{2.8}$$

where $E(x,y)$ and $F(x,y)$ are some functions; $R^\mu_\sigma, S^\alpha_{s\sigma}, T^s_{\alpha\sigma}, A^\alpha_{s\sigma}$ and $B^s_{\alpha\sigma}$ are linear differential operators. We have

$$\begin{aligned} &R^\mu_\sigma=r^{\mu\nu(k)}_\sigma\partial_{\nu(k)}, \quad S^\alpha_{s\sigma}=s^{\alpha\nu(l)}_{s\sigma}\partial_{\nu(l)},\\ &T^s_{\alpha\sigma}=t^{s\nu(m)}_{\alpha\sigma}\partial_{\nu(m)}, \quad A^\alpha_{s\sigma}=a^{\alpha\nu(n)}_{s\sigma}\partial_{\nu(n)},\\ &B^s_{\alpha\sigma}=b^{s\nu(p)}_{\alpha\sigma}\partial_{\nu(p)}, \quad r^{\mu\nu(k)}_\sigma=r^{\overbrace{\mu\nu\cdots\lambda}^{k}}_\sigma,\\ &\partial_{\nu(k)}=\overbrace{\partial_\nu\cdots\partial_\lambda}^{k}, \quad \text{etc.}, \end{aligned} \tag{2.9}$$

where $r^{\mu\nu(k)}_\sigma, s^{\alpha\nu(l)}_{s\sigma}, t^{s\nu(m)}_{\alpha\sigma}, b^{s\nu(p)}_{\alpha\sigma}$, are some functions of x, $\varphi^\alpha_{(s)}$ and $\pi^{(s)}_\alpha$ (for example, $R^\mu_\sigma=r^{\mu\nu\cdots\lambda}_\sigma\partial_\nu\cdots\partial_\lambda$, etc.) and $\varepsilon^\alpha(x)$ $(\alpha=1,2,\cdots,r)$ are arbitrary infinitesimal independent functions; their values and derivatives up to required order vanish on the boundary of the space-time domain. The Jacobian of the transformation of the canonical variables in (2.8) is denoted by $\bar{J}[\varphi^\alpha_{(s)},\pi^{(s)}_\alpha,\varepsilon]=1+J_1[\varphi^\alpha_{(s)},\pi^{(s)}_\alpha,\varepsilon]$, where J_1 is also an infinitesimal quantity.

It is supposed that the variation of the canonical action under the transformation (2.8) is given by

$$\Delta I^{\mathrm{P}} = \int \mathrm{d}^4 x U_\sigma \varepsilon^\sigma(x), \tag{2.10}$$

where U_σ are linear differential operators, $U_\sigma = \mu_\sigma^{\gamma(q)} \partial_{\gamma(q)}$, $\mu_\sigma^{\gamma(q)}$ are some functions of $x, \varphi_{(s)}^\alpha$ and $\pi_\alpha^{(s)}$. Under the transformation (2.8), from the expression (2.6) of the phase-space generating functional and (2.10), one gets [9]

$$\begin{aligned}
&\int \mathcal{D}\varphi_{(s)}^\alpha \mathcal{D}\pi_\alpha^{(s)} \Bigg(1 + J_1 + \mathrm{i}\Delta I^{\mathrm{P}} \\
&\quad + \mathrm{i} \int \mathrm{d}^4 x \Big\{ J_\alpha^s \delta\varphi_{(s)}^\alpha + K_s^\alpha \delta\pi_\alpha^{(s)} \\
&\quad + \partial_\mu [(J_\alpha^s \varphi_{(s)}^\alpha + K_s^\alpha \pi_\alpha^{(s)}) \Delta x^\mu] \Big\} \Bigg) \\
&\times \exp\left\{ \mathrm{i} \int \mathrm{d}^4 x (\mathcal{L}^{\mathrm{P}} + J_\alpha^s \varphi_{(s)}^\alpha + K_s^\alpha \pi_\alpha^{(s)}) \right\} \\
&= \int \mathcal{D}\varphi_{(s)}^\alpha \mathcal{D}\pi_\alpha^{(s)} \Bigg(1 + J_1 \\
&\quad + \mathrm{i} \int \mathrm{d}^4 x \Big\{ U_\sigma \varepsilon^\sigma(x) + J_\alpha^s \delta\varphi_{(s)}^\alpha + K_s^\alpha \delta\pi_\alpha^{(s)} \\
&\quad + \partial_\mu [(J_\alpha^s \varphi_{(s)}^\alpha + K_s^\alpha \pi_\alpha^{(s)}) \Delta x^\mu] \Big\} \Bigg) \\
&\times \exp\left\{ \mathrm{i} \int \mathrm{d}^4 x (\mathcal{L}^{\mathrm{P}} + J_\alpha^s \varphi_{(s)}^\alpha + K_s^\alpha \pi_\alpha^{(s)}) \right\},
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
\Delta I^{\mathrm{P}} &= \int \mathrm{d}^4 x \left\{ \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha} \delta\varphi_{(s)}^\alpha + \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}} \delta\pi_\alpha^{(s)} \right. \\
&\quad \left. + D(\pi_\alpha^{(s)} \delta\varphi_{(s)}^\alpha) + \partial_\mu \left[\left(\pi_\alpha^{(s)} \varphi_{(s+1)}^\alpha - \mathcal{H}_{\mathrm{c}} \right) \Delta x^\mu \right] \right\},
\end{aligned} \tag{2.12}$$

$$\frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha} = -\dot{\pi}_\alpha^{(s)} - \frac{\delta H_{\mathrm{c}}}{\delta \varphi_{(s)}^\alpha}, \quad \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}} = \dot{\varphi}_{(s)}^\alpha - \frac{\delta H_{\mathrm{c}}}{\delta \pi_\alpha^{(s)}}, \tag{2.13}$$

$$\begin{aligned}
\delta\varphi_{(s)}^\alpha &= \Delta\varphi_{(s)}^\alpha - \varphi_{(s),\mu}^\alpha \Delta x^\mu, \\
\delta\pi_\alpha^{(s)} &= \Delta\pi_\alpha^{(s)} - \pi_{\alpha,\mu}^{(s)} \Delta x^\mu;
\end{aligned} \tag{2.14}$$

$D = \mathrm{d}/\mathrm{d}t$ and H_{c} is the generalized canonical Hamiltonian (2.5), and $\mathcal{H}_{\mathrm{c}}$ is the generalized canonical Hamiltoni density. According to the boundary conditions of the functions $\varepsilon^\sigma(x)$, from (2.11) and (2.12), we have

$$\begin{aligned}
&\int \mathcal{D}\varphi_{(s)}^\alpha \mathcal{D}\pi_\alpha^{(s)} \left\{ \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha} \delta\varphi_{(s)}^\alpha + \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}} \delta\pi_\alpha^{(s)} \right. \\
&\quad \left. + D\left(\pi_\alpha^{(s)} \delta\varphi_{(s)}^\alpha \right) - U_\sigma \varepsilon^\sigma(x) \right\} \\
&\times \exp\left\{ \mathrm{i} \int \mathrm{d}^4 x (\mathcal{L}^{\mathrm{P}} + J_\alpha^s \varphi_{(s)}^\alpha + K_s^\alpha \pi_\alpha^{(s)}) \right\} = 0.
\end{aligned} \tag{2.15}$$

Substituting (2.8) and (2.14) into (2.15), and integrating by parts the corresponding terms in (2.15), then functionally differentiating the obtained results with respect to $\varepsilon^\sigma(x)$, and using the boundary conditions of the functions $\varepsilon^\sigma(x)$, one gets

$$\begin{aligned}
&\int \mathcal{D}\varphi_{(s)}^\alpha \mathcal{D}\pi_\alpha^{(s)} \left\{ \tilde{S}_{s\sigma}^\alpha(x) \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha(x)} + \tilde{T}_{\alpha\sigma}^s(x) \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}(x)} \right. \\
&- \tilde{R}_\sigma^\mu(x) \left[\varphi_{(s),\mu}^\alpha \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha(x)} + \pi_{\alpha,\mu}^{(s)} \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}(x)} \right] \\
&+ \int \mathrm{d}^4 y \left[\tilde{A}_{s\sigma}^\alpha(y) (E(y,x) \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha(x)} + D(\pi_\alpha^{(s)}(x) E(y,x)) \right. \\
&\quad \left. \left. + \tilde{B}_{\alpha\sigma}^s(y) \left(F(y,x) \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}(y)} \right) \right] - \tilde{U}_\sigma(1) \right\} \\
&\times \exp\left\{ \mathrm{i} \int \mathrm{d}^4 x (\mathcal{L}^{\mathrm{P}} + J_\alpha^s \varphi_{(s)}^\alpha + K_s^\alpha \pi_\alpha^{(s)}) \right\} = 0,
\end{aligned} \tag{2.16}$$

where $\tilde{S}_{s\sigma}^\alpha, \tilde{T}_{\alpha\sigma}^s, \tilde{R}_\sigma^\mu, \tilde{A}_{s\sigma}^\alpha, \tilde{B}_{\alpha\sigma}^s$ and $\tilde{U}_\sigma$ are adjoint operators with respect to $S_{s\sigma}^\alpha, T_{\alpha\sigma}^s, R_\sigma^\mu, A_{s\sigma}^\alpha, B_{\alpha\sigma}^s$ and U_σ, respectively [18]; for example,

$$\int f R_\sigma^\mu g \mathrm{d}^4 x = \int g \tilde{R}_\sigma^\mu f \mathrm{d}^4 x + [\cdot]_{\mathrm{B}},$$

where $[\cdot]_{\mathrm{B}}$ stands for boundary terms.

Functionally differentiating (2.16) with respect to $J_\alpha^0(x)$ a total of n times, one obtains

$$\begin{aligned}
&\int \mathcal{D}\varphi_{(s)}^\alpha \mathcal{D}\pi_\alpha^{(s)} \left\{ \tilde{S}_{s\sigma}^\alpha(x) \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha(x)} + \tilde{T}_{\alpha\sigma}^s(x) \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}(x)} \right. \\
&- \tilde{R}_\sigma^\mu(x) \left[\varphi_{(s),\mu}^\alpha \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha}(x) + \pi_{\alpha,\mu}^{(s)} \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}(x)} \right] \\
&+ \int \mathrm{d}^4 y \left[\tilde{A}_{s\sigma}^\alpha(y) (E(y,x) \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha(y)} \right. \\
&\quad + D(\pi_\alpha^{(s)}(y) E(y,x)) \\
&\quad \left. \left. + \tilde{B}_{\alpha\sigma}^s(y) \left(F(y,x) \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}(y)} \right) \right] - \tilde{U}_\sigma(1) \right\} \\
&\times \varphi^\alpha(x_1) \varphi^\alpha(x_2) \cdots \varphi^\alpha(x_n) \\
&\times \exp\left\{ \mathrm{i} \int \mathrm{d}^4 x (\mathcal{L}^{\mathrm{P}} + J_\alpha^s \varphi_{(s)}^\alpha + K_s^\alpha \pi_\alpha^{(s)}) \right\} = 0.
\end{aligned} \tag{2.17}$$

Let $J_\alpha^s = K_s^\alpha = 0$ in (2.17); then one gets

$$\begin{aligned}
&\langle 0 | T^* \left\{ \tilde{S}_{s\sigma}^\alpha(x) \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha} + \tilde{T}_{\alpha\sigma}^s(x) \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}} \right. \\
&- \tilde{R}_\sigma^\mu(x) \left[\varphi_{(s),\mu}^\alpha \frac{\delta I^p}{\delta \varphi_{(s)}^\alpha} + \pi_{\alpha,\mu}^{(s)} \frac{\delta I^{\mathrm{P}}}{\delta \pi_\alpha^{(s)}} \right] \\
&+ \int \mathrm{d}^4 y \left[\tilde{A}_{s\sigma}^\alpha(y) (E(y,x) \cdot \frac{\delta I^{\mathrm{P}}}{\delta \varphi_{(s)}^\alpha} \right.
\end{aligned}$$

$$+D\left(\pi_\alpha^{(s)}(y)E(y,x)\right)$$
$$+\ \tilde{B}^s_{\alpha\sigma}(y)\left(F(y,x)\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}}\right)\Bigg] - \tilde{U}_\sigma(1)\Bigg\}$$
$$\times\ \varphi^\alpha(x_1)\varphi^\alpha(x_2)\cdots\varphi^\alpha(x_n)|0\rangle = 0, \qquad (2.18)$$

where the symbol T^* stands for the covariantized T-product [19] in which derivatives of operators inside a T-product are defined in terms of the formula

$$\langle 0|T^*[\partial_\mu\varphi(x)\partial_\nu\varphi(y)\cdots]|0\rangle$$
$$= \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\langle 0|T[\varphi(x)\varphi(y)\cdots]|0\rangle,$$

and $|0\rangle$ is the vacuum state of the fields. Fixing t and letting

$$t_1, t_2, \cdots, t_m \longrightarrow +\infty, \quad t_{m+1}, t_{m+2}, \cdots, t_n \longrightarrow -\infty,$$

noting that $\varphi^\alpha(\vec{x},-\infty)|\rangle = |\mathrm{in}\rangle$, $\langle 0|\varphi^\alpha(\vec{x},\infty) = \langle\mathrm{out}|$ and using the reduction formula [19], one can write the expressions (2.18) as

$$\langle\mathrm{out}, m|\Bigg\{\tilde{S}^\alpha_{s\sigma}(x)\frac{\delta I^{\mathrm{P}}}{\delta\varphi^\alpha_{(s)}} + \tilde{T}^s_{\alpha\sigma}\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}}$$
$$-\ \tilde{R}^\mu_\sigma(x)\left[\varphi^\alpha_{(s),\mu}\frac{\delta I^p}{\delta\varphi^\alpha_{(s)}} + \pi^{(s)}_{\alpha,\mu}\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}}\right]$$
$$+\int \mathrm{d}^4y\Bigg[\tilde{A}^\alpha_{s\sigma}(y)(E(y,x)\frac{\delta I^{\mathrm{P}}}{\delta\varphi^\alpha_{(s)}} + D\left(\pi_\alpha^{(s)}(y)E(y,x)\right)$$
$$+\ \tilde{B}^s_{\alpha\sigma}(y)\left(F(y,x)\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}}\right)\Bigg] - \tilde{U}_\sigma(1)\Bigg\}$$
$$\times|n-m,\mathrm{in}\rangle = 0. \qquad (2.19)$$

Since m and n are arbitrary, one obtains

$$\tilde{S}^\alpha_{s\sigma}(x)\frac{\delta I^{\mathrm{P}}}{\delta\varphi^\alpha_{(s)}} + \tilde{T}^s_{\alpha\sigma}\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}}$$
$$-\ \tilde{R}^\mu_\sigma(x)\left[\varphi^\alpha_{(s),\mu}\frac{\delta I^{\mathrm{P}}}{\delta\varphi^\alpha_{(s)}} + \pi^{(s)}_{\alpha,\mu}\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}}\right]$$
$$+\int \mathrm{d}^4y\Bigg[\tilde{A}^\alpha_{s\sigma}(y)\Bigg(E(y,x)\frac{\delta I^{\mathrm{P}}}{\delta\varphi^\alpha_{(s)}}$$
$$+\ D(\pi_\alpha^{(s)}(y)E(y,x))\Bigg)$$
$$+\ \tilde{B}^s_{\alpha\sigma}(y)\left(F(y,x)\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}}\right)\Bigg] - \tilde{U}_\sigma(1)\Bigg\} = 0. \quad (2.20)$$

These expressions, (2.20), are called quantum canonical NIs under the local and non-local transformation (2.8) for a system with a regular higher-order Lagrangian. For the case the transformation (2.8) is only a local one ($E = F = 0$), the expressions (2.20) can be written as

$$\tilde{S}^\alpha_{s\sigma}(x)\frac{\delta I^{\mathrm{P}}}{\delta\varphi^\alpha_{(s)}} + \tilde{T}^s_{\alpha\sigma}\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}} \qquad (2.21)$$
$$-\ \tilde{R}^\mu_\sigma(x)\left[\varphi^\alpha_{(s),\mu}\frac{\delta I^{\mathrm{P}}}{\delta\varphi^\alpha_{(s)}} + \pi^{(s)}_{\alpha,\mu}\frac{\delta I^{\mathrm{P}}}{\delta\pi_\alpha^{(s)}}\right] - \tilde{U}_\sigma(1) = 0.$$

These expressions coincide with the classical canonical NIs [4] whether the Jacobian of the transformation (2.8) is equal to unity or not.

Let us now consider a system with a singular higher-order Lagrangian $\mathcal{L}(\varphi^\alpha, \varphi^\alpha_{.u}, \cdots, \varphi^\alpha_{.u(N)})$, whose generalized Hessian matrix $(H_{\alpha\beta})$ is degenerate:

$$\det|H_{\alpha\beta}| = \det\left|\frac{\partial^2\mathcal{L}}{\partial\varphi^\alpha_{(N)}\partial\varphi^\beta_{(N)}}\right| = 0; \qquad (2.22)$$

hence one cannot solve all $\varphi^\alpha_{(N)}$ from (2.2). Then there are some constraints among the canonical variable in phase space [9,20]:

$$\Phi^0_a(\varphi^\alpha_{(s)}, \pi_\alpha^{(s)}) \approx 0 \quad (a = 1, 2, \cdots, n-R), \qquad (2.23)$$

where the sign $\approx$ represents a weak equality, and the rank of the generalized Hessian matrix is assumed to be R. Equations (2.23) are called primary constraints. From the stationarity of the constraint, one can define successively the secondary constraints from the primary ones. This process of consistency requirements will terminate at some stage when new constraints no longer appear. All the constraints are classified into two classes. The constraints in the first class are those whose generalized Poisson brackets with any of the constraints are equal to zero or equal to a linear combination of the constraints; if this is not the case, the constraints are called second class. Let $\Lambda_k(\varphi^\alpha_{(s)}, \pi_\alpha^{(s)}) \approx 0$ $(k = 1, 2, \cdots, K_1)$ be first-class constraints, and $\theta_i(\varphi^\alpha_{(s)}\pi_\alpha^{(s)}) \approx 0$ $(i = 1, 2, \cdots, I_1)$ second-class constraints. The path-integral quantization in phase space for this system can be formulated by using the Batalin–Fradkin–Vilkovisky (BFV) scheme [21] or the Faddeev–Senjanovic (FS) scheme [22], but the latter is more convenient. This is according to the FS path-integral quantization scheme [22], and the gauge conditions connected with the first-class constraints are denoted by $\Omega_k(\varphi^\alpha_{(s)}, \pi_\alpha^{(s)}) \approx 0$. The phase-space generating functional of the Green function for this system with a singular higher-order Lagrangian can be written as [9,23]

$$Z[J,K]$$
$$= \int \mathcal{D}\varphi^\alpha_{(s)}\mathcal{D}\pi_\alpha^{(s)}\prod_{i,k,l}\delta(\theta_i)\delta(\Lambda_k)\delta(\Omega_l)$$
$$\times\ \det|\{\Lambda_k, \Omega_l\}|\cdot[\det|\{\theta_i,\theta_j\}|]^{1/2}$$
$$\times\ \exp\left\{\mathrm{i}\int \mathrm{d}^4x(\mathcal{L}^{\mathrm{P}} + J^s_\alpha\varphi^\alpha_{(s)} + K^\alpha_s\pi_\alpha^{(s)})\right\}, \qquad (2.24)$$

where the symbol $\{\cdot,\cdot\}$ represents the generalized Poisson bracket. Using the δ-function and the integral properties of the Grassmann variables $C_a(x)$ and $\bar{C}_b(x)$, one can write (2.24) as [9]

$$Z[J,K,\eta,\bar{j},\bar{k},j,k]$$

$$= \int \mathcal{D}\varphi^{\alpha}_{(s)} \mathcal{D}\pi^{(s)}_{\alpha} \mathcal{D}\lambda_m \mathcal{D}C_a \mathcal{D}\bar{\pi}^a \mathcal{D}\bar{C}_a \mathcal{D}\pi^a \times \exp\left\{ \mathrm{i} \int \mathrm{d}^4x \Big(\mathcal{L}^{\mathrm{P}}_{\mathrm{eff}} + J^s_{\alpha}\varphi^{\alpha}_{(s)} + K^{\alpha}_s \pi^{(s)}_{\alpha} + \eta^m\lambda_m + \bar{j}^a C_a + \bar{k}_a \pi^a + \bar{C}_a j^a + \bar{\pi}^a k_a \Big) \right\}, \tag{2.25}$$

where

$$\mathcal{L}^{\mathrm{P}}_{\mathrm{eff}} = \mathcal{L}^{\mathrm{P}} + \mathcal{L}_m + \mathcal{L}_{\mathrm{gh}}, \tag{2.26}$$

$$\mathcal{L}_m = \lambda_k \Lambda_k + \lambda_l \Omega_l + \lambda_i \theta_i, \tag{2.27}$$

$$\mathcal{L}_{\mathrm{gh}} = \int \mathrm{d}^4x \left[\bar{C}_k(x)\{\Lambda_k(x), \Omega_l(y)\} C_l(y) + \frac{1}{2}\bar{C}_i(x)\{\theta_i(x), \theta_j(y)\}\theta_j(y) \right], \tag{2.28}$$

and $\lambda_m = (\lambda_k, \lambda_l, \lambda_i)$; $\bar{\pi}^a(x)$ and $\pi^b(x)$ are canonical momenta conjugate to $C_a(x)$ and $\bar{C}_b(x)$, respectively. $\eta^m, \bar{j}^a$, $\bar{k}_a, j_{\underline{a}}$ and k_a are exterior sources with respect to λ^m, C_a, $\pi^a, \bar{C}_a$ and $\bar{\pi}^a$, respectively. For the sake of simplicity, let us denote $\varphi^{\alpha}_{(s)} = (\varphi^{\alpha}_{(s)}, \lambda_m, C_a, \bar{C}_a), \pi^{(s)}_{\alpha} = (\pi^{(s)}_{\alpha}, \bar{\pi}^a, \pi^a)$, $J^s = (J^s_{\alpha}, \eta^m, j^a, \bar{j}^a)$ and $K^{\alpha}_s = (K^{\alpha}_s, k_a, \bar{k}_a)$; thus the expression (2.25) can be written as

$$Z[J,K] = \int \mathcal{D}\varphi^{\alpha}_{(s)} \mathcal{D}\pi^{(s)}_{\alpha} \times \exp\left\{ \mathrm{i} \int \mathrm{d}^4x (\mathcal{L}^{\mathrm{P}}_{\mathrm{eff}} + J^s_{\alpha}\varphi^{\alpha}_{(s)} + K^{\alpha}_s \pi^{(s)}_{\alpha}) \right\}. \tag{2.29}$$

For a system with a singular higher-order Lagrangian, one can proceed in the same way as for a system with a regular higher-order Lagrangian to derive the quantum canonical NIs under a local and non-local transformation in extended phase space. The results of the quantum canonical NIs for a singular higher-order Lagrangian differ from the classical ones [4] in that one must use an effective canonical action $I^{\mathrm{P}}_{\mathrm{eff}}$ instead of I^{P} in the corresponding expressions.

Thus, we obtain the identity relations (2.20) and (2.21) between the functional derivatives and their derivatives at the quantum level, and this leads to a reduction in the number of independent functional derivatives $\frac{\delta I^{\mathrm{P}}_{\mathrm{eff}}}{\delta \varphi^{\alpha}_{(s)}}$ and $\frac{\delta I^{\mathrm{P}}_{\mathrm{eff}}}{\delta \pi^{(s)}_{\alpha}}$.

3 Gauge-invariant system

For a system with a gauge-invariant Lagrangian L involving higher-order derivatives of the field variables, the effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$ in configuration space can be found by using the FP trick through a transformation of the path (functional) integral [17]: $\mathcal{L}^{\mathrm{P}}_{\mathrm{eff}} = \mathcal{L} + \mathcal{L}_f + \mathcal{L}_{\mathrm{gh}}$, where $\mathcal{L}_f$ is determined by the gauge conditions and $\mathcal{L}_{\mathrm{gh}}$ is a ghost term. The configuration-space generating functional of the Green function for this system can be written as [9]

$$Z[J] = \int \mathcal{D}\varphi \exp\left\{ \mathrm{i} \int \mathrm{d}^4x (\mathcal{L}_{\mathrm{eff}} + J\varphi) \right\}, \tag{3.1}$$

where φ represents all field variables, and J is an exterior source with respect to φ.

Let us now consider an infinitesimal local and non-local transformation in the configuration space:

$$\begin{cases} x^{\mu'} = x^{\mu} + \Delta x^{\mu} = x^{\mu} + R^{\mu}_{\sigma}\varepsilon^{\sigma}(x), \\ \varphi'(x') = \varphi(x) + \Delta\varphi(x) \\ \quad = \varphi(x) + S_{\sigma}\varepsilon^{\sigma}(x) + \int \mathrm{d}^4y E(x,y) N_{\sigma}(y)\varepsilon^{\sigma}(y), \end{cases} \tag{3.2}$$

where $R^{\mu}_{\sigma}, S_{\sigma}$ and N_{σ} are linear differential operators, and $\varepsilon^{\sigma}(x)$ $(\sigma = 1, 2, \cdots, r)$ are arbitrary infinitesimal independent functions; the values and their derivatives up to required order on the boundary of the space-time domain vanish. It is supposed that the variation of the effective action under the transformation (3.2) is given by

$$\Delta I_{\mathrm{eff}} = \Delta \int \mathrm{d}^4x \mathcal{L}_{\mathrm{eff}} = \int \mathrm{d}^4x V_{\sigma}\varepsilon^{\sigma}(x), \tag{3.3}$$

where V_{σ} are some linear differential operators. The Jacobian of the transformation (3.2) of the field variables is denoted by $\bar{J}^c = 1 + J^c_1[\varphi, \varepsilon]$. Under the transformation (3.2), the configuration-space generating functional (3.1) becomes

$$Z[J] = \int \mathcal{D}\varphi \left\{ 1 + J^c_1 + \mathrm{i}\Delta I_{\mathrm{eff}} + \mathrm{i}\int \mathrm{d}^4x [J\delta\varphi + \partial_{\mu}(J\varphi\Delta x^{\mu})] \right\} \times \exp\left\{ \mathrm{i} \int \mathrm{d}^4x (\mathcal{L}_{\mathrm{eff}} + J\varphi) \right\}, \tag{3.4}$$

where [9]

$$\Delta I_{\mathrm{eff}} = \int \mathrm{d}^4x \left\{ \frac{\delta I_{\mathrm{eff}}}{\delta\varphi} \left[(S_{\sigma} - \varphi_{,\mu} R^{\mu}_{\sigma})\varepsilon^{\sigma}(x) + \int \mathrm{d}^4y E(x,y) N_{\sigma}(y)\varepsilon^{\sigma}(y) \right] + \partial_{\mu}\left(j^{\mu}_{\sigma}\varepsilon^{\sigma}(x) \right) + \partial_{\mu}\left[\sum_{m=0}^{N-1} \prod_{\mathrm{eff}}^{\mu\nu(m)} \partial_{\nu(m)} \int \mathrm{d}^4y E(x,y) N_{\sigma}(y)\varepsilon^{\sigma}(y) \right] \right\}, \tag{3.5}$$

$$\delta I_{\mathrm{eff}}/\delta\varphi = (-1)^m \partial_{\mu(m)} \mathcal{L}^{\mu(m)}_{\mathrm{eff}}, \tag{3.6}$$

$$\mathcal{L}^{\mu(m)}_{\mathrm{eff}} = \frac{1}{m!} \sum_{\substack{\text{all permutations of indices} \\ \mu(m)}} \frac{\partial \mathcal{L}_{\mathrm{eff}}}{\partial \varphi_{,\mu(m)}}, \tag{3.7}$$

$$j^{\mu}_{\sigma} = \mathcal{L}_{\mathrm{eff}} R^{\mu}_{\sigma} + \sum_{m=0}^{N-1} \prod_{\mathrm{eff}}^{\mu\nu(m)} \partial_{\nu(m)}(S_{\sigma} - \varphi_{,\mu} R^{\mu}_{\sigma}), \tag{3.8}$$

$$\prod_{\mathrm{eff}}^{\mu\nu(m)} = \sum_{l=0}^{N-(m+1)} (-1)^l \partial_{\lambda(l)} \mathcal{L}^{\mu\nu(m)\lambda(l)}_{\mathrm{eff}}. \tag{3.9}$$

According to the boundary conditions of the $\varepsilon^\sigma(x)$ from (3.2)–(3.9), one gets

$$\int \mathcal{D}\varphi \int \mathrm{d}^4x \left\{ \frac{\delta I_{\text{eff}}}{\delta\varphi} \left[(S_\sigma - \varphi_{,\mu} R^\mu_\sigma)\varepsilon^\sigma(x) + \int \mathrm{d}^4y E(x,y) N_\sigma(y)\varepsilon^\sigma(y) \right] - V_\sigma \varepsilon^\sigma(x) + \partial_\mu \left[\sum_{m=0}^{N-1} \prod_{\text{eff}}^{\mu\nu(m)} \partial_{\nu(m)} \int \mathrm{d}^4y E(x,y) N_\sigma(y) \varepsilon^\sigma(y) \right] \right\} \times \exp\left\{ \mathrm{i} \int \mathrm{d}^4x (\mathcal{L}_{\text{eff}} + J\varphi) \right\} = 0. \tag{3.10}$$

Integrating by parts of the terms connected with the differential operators $S_\sigma, R^\mu_\sigma, N_\sigma$ and V_σ in the expression (3.10), and appealing to the arbitrariness of the $\varepsilon^\sigma(x)$, one can force the boundary terms to vanish; after this one can functionally differentiate the results with respect to $\varepsilon^\sigma(x)$. Then one obtains

$$\int \mathcal{D}\varphi \int \mathrm{d}^4x \left\{ \tilde{S}_\sigma(x) \frac{\delta I_{\text{eff}}}{\delta\varphi} - \tilde{R}^\mu_\sigma \left(\varphi_{,\mu}(x) \frac{\delta I_{\text{eff}}}{\delta\varphi} \right) + \int \mathrm{d}^4y \tilde{N}_\sigma(y) \left[E(x,y) \frac{\delta I_{\text{eff}}}{\delta\varphi} + \partial_\mu \left(\sum_{m=0}^{N-1} \prod_{\text{eff}}^{\mu\nu(m)} \partial_{\nu(m)} E(x,y) \right) \right] - \tilde{V}_\sigma(1) \right\} \times \exp\left\{ \mathrm{i} \int \mathrm{d}^4x (\mathcal{L}_{\text{eff}} + J\varphi) \right\} = 0, \tag{3.11}$$

where $\tilde{S}_\sigma, \tilde{R}^\mu_\sigma, \tilde{N}_\sigma$ and $\tilde{V}_\sigma$ are adjoint operators with respect to $S_\sigma, R^\mu_\sigma, N_\sigma$ and V_σ, respectively [18]. Functionally differentiating (3.11) with respect to $J(x)$ n times, and letting $J = 0$; fixing t and letting $t_1, t_2, \cdots, t_m \longrightarrow -\infty$, $t_{m+1}, t_{m+2}, \cdots, t_n \longrightarrow \infty$, one can proceed in the same way as discussed in Sect. 2, to obtain

$$\tilde{S}_\sigma(x) \frac{\delta I_{\text{eff}}}{\delta\varphi} - \tilde{R}^\mu_\sigma \left(\varphi_{,\mu}(x) \frac{\delta I_{\text{eff}}}{\delta\varphi} \right) + \int \mathrm{d}^4y \tilde{N}_\sigma(y) \left[E(x,y) \frac{\delta I_{\text{eff}}}{\delta\varphi} + \partial_\mu \left(\sum_{m=0}^{N-1} \prod_{\text{eff}}^{\mu\nu(m)} \partial_{\nu(m)} E(x,y) \right) \right] - \tilde{V}_\sigma(1) = 0. \tag{3.12}$$

These expressions are called quantum NIs under a local and non-local transformation in configuration space for a gauge-invariant system with a higher-order Lagrangian. For the case the transformation (3.2) is only a local one ($E = 0$ in (3.2)), from (3.12), one has

$$\tilde{S}_\sigma(x) \frac{\delta I_{\text{eff}}}{\delta\varphi} - \tilde{R}^\mu_\sigma \left(\varphi_{,\mu}(x) \frac{\delta I_{\text{eff}}}{\delta\varphi} \right) - \tilde{V}_\sigma(1) = 0. \tag{3.13}$$

These quantum NIs also hold true whether the Jacobian of the transformation is equal to unity or not. The identities (3.12) and (3.13) differ from the classical one in that the action in quantum NIs is an effective action I_{eff} but not a classical action I.

4 Quantum conservation laws

Based on the quantum NIs, for certain cases one can obtain quantum strong conservation laws for a system with higher-order Lagrangian; this result holds true whether the equations of motion at the quantum level are satisfied or not. Using the quantum equations of motion, one can get quantum (weak) conservation laws.

For the sake of simplicity, we consider a local transformation in (2.8) with $\Delta x^u = 0$, and

$$\begin{cases} S^\alpha_{s\sigma} = a^\alpha_{s\sigma} + a^{\alpha\mu}_{s\sigma}\partial_\mu + a^{\alpha\mu\nu}_{s\sigma}\partial_\mu\partial_\nu, \\ T^s_{\alpha\sigma} = b^s_{\alpha\sigma} + b^{s\mu}_{\alpha\sigma}\partial_\mu + b^{s\mu\nu}_{\alpha\sigma}\partial_\mu\partial_\nu, \end{cases} \tag{4.1}$$

where $a^\alpha_{s\sigma}, a^{\alpha\mu}_{s\sigma}, a^{\alpha\mu\nu}_{s\sigma}, b^s_{\alpha\sigma}, b^{s\mu}_{\alpha\sigma}$ and $b^{s\mu\nu}_{\alpha\sigma}$ are some smoothed functions of $x, \varphi^\alpha_{(s)}$ and $\pi^{(s)}_\alpha$. Under the transformation (2.8) with (4.1), it is supposed that the change of the effective canonical Lagrangian $\mathcal{L}^{\mathrm{P}}_{\text{eff}}$ is given by

$$\delta\mathcal{L}^{\mathrm{P}}_{\text{eff}} = U_\sigma\varepsilon^\sigma(x) = (u_\sigma + u^\mu_\sigma\partial_\mu + u^{\mu\nu}_\sigma\partial_\mu\partial_\nu)\varepsilon^\sigma(x), \tag{4.2}$$

where $u_\sigma, u^\mu_\sigma, u^{\mu\nu}_\sigma$ are some smoothed functions of $x, \varphi^\alpha_{(s)}$ and $\pi^{(s)}_\alpha$. The quantum canonical NIs (2.21) in this case become

$$a^\alpha_{s\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\varphi^\alpha_{(s)}} - \partial_\mu\left(a^{\alpha\mu}_{s\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\varphi^\alpha_{(s)}}\right) + \partial_\mu\partial_\nu\left(a^{\alpha\mu\nu}_{s\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\varphi^\alpha_{(s)}}\right) + b^s_{\alpha\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\pi^{(s)}_\alpha} - \partial_\mu\left(b^{s\mu}_{\alpha\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\pi^{(s)}_\alpha}\right) + \partial_\mu\partial_\nu\left(b^{s\mu\nu}_{\alpha\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\pi^{(s)}_\alpha}\right) = u_\sigma - \partial_\mu u^\mu_\sigma + \partial_\mu\partial_\nu u^{\mu\nu}_\sigma. \tag{4.3}$$

Under the transformation (2.8) with (4.1), from the variation of the effective canonical action, one has the basic identity

$$\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\varphi^\alpha_{(s)}}(a^\alpha_{s\sigma} + a^{\alpha\mu}_{s\sigma}\partial_\mu + a^{\alpha\mu\nu}_{s\sigma}\partial_\mu\partial_\nu)\varepsilon^\sigma(x) + \frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\pi^{(s)}_\alpha}(b^s_{\alpha\sigma} + b^{s\mu}_{\alpha\sigma}\partial_\mu + b^{s\mu\nu}_{\alpha\sigma}\partial_\mu\partial_\nu)\varepsilon^\sigma(x) + D[\pi^{(s)}_\alpha S^\alpha_{s\sigma}\varepsilon^\sigma(x)] = (u_\sigma + u^\mu_\sigma\partial_\mu + u^{\mu\nu}_\sigma\partial_\mu\partial_\nu)\varepsilon^\sigma(x). \tag{4.4}$$

Multiplying the identities (4.3) by the $\varepsilon^\sigma(x)$ and summing the index σ from 1 to r, and subtracting the obtained result from the basic identity (4.4), if the index μ, ν of the coefficients $a^{\alpha\mu\nu}_{s\sigma}, b^{s\mu\nu}_{\alpha\sigma}$ are symmetrical, one obtains

$$\partial_\mu\left\{\left[a^{\alpha\mu}_{s\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\varphi^\alpha_{(s)}} + b^{s\mu}_{\alpha\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\pi^{(s)}_\alpha} + \partial_\nu\left(a^{\alpha\mu\nu}_{s\sigma}\frac{\delta I^{\mathrm{P}}_{\text{eff}}}{\delta\varphi^\alpha_{(s)}}\right)\right.\right.$$

$$- \left(a^{\alpha\mu\nu}_{s\sigma}\frac{\delta I^{\rm P}_{\rm eff}}{\delta\varphi^{\alpha}_{(s)}}\right)\partial_\nu + \partial_\nu\left(b^{s\mu\nu}_{\alpha\sigma}\frac{\delta I^{\rm P}_{\rm eff}}{\delta\pi^{(s)}_{\alpha}}\right) - \left(b^{s\mu\nu}_{\alpha\sigma}\frac{\delta I^{\rm P}_{\rm eff}}{\delta\pi^{(s)}_{\alpha}}\right)\partial_\nu - u^{\mu}_{\sigma} + \partial_\nu u^{\mu\nu}_{\sigma} - u^{\mu\nu}_{\sigma}\partial_\nu + D(\pi^{(s)}_{\alpha}S^{\alpha}_{s\sigma})\Bigg] \times \varepsilon^{\sigma}(x)\} = 0. \tag{4.5}$$

Taking the integral of the identity (4.5) on a $t = \text{const}$ space-like hypersurface, one gets the strong conservation laws at the quantum level

$$Q = \int \mathrm{d}^3x j_\sigma \varepsilon^\sigma(x) = \text{const} \quad (\sigma = 1, 2, \cdots, r), \tag{4.6a}$$

where

$$j_\sigma = a^{\alpha 0}_{s\sigma}\frac{\delta I^{\rm P}_{\rm eff}}{\delta\varphi^{\alpha}_{(s)}} + b^{s0}_{\alpha\sigma}\frac{\delta I^{\rm P}_{\rm eff}}{\delta\pi^{(s)}_{\alpha}} + \partial_\nu\left(a^{\alpha 0\nu}_{s\sigma}\frac{\delta I^{\rm P}_{\rm eff}}{\delta\varphi^{\alpha}_{(s)}}\right) - \left(a^{\alpha 0\nu}_{s\sigma}\frac{\delta I^{p}_{\rm eff}}{\delta\varphi^{\alpha}_{(s)}}\right)\partial_\nu + \partial_\nu\left(b^{s0\nu}_{\alpha\sigma}\frac{\delta I^{\rm P}_{\rm eff}}{\delta\pi^{(s)}_{\alpha}}\right) - \left(b^{s0\nu}_{\alpha\sigma}\frac{\delta I^{\rm P}_{\rm eff}}{\delta\pi^{(s)}_{\alpha}}\right)\partial_\nu - u^{0}_{\sigma} + \partial_\nu u^{0\nu}_{\sigma} - u^{0\nu}_{\sigma}\partial_\nu + \pi^{(s)}_{\alpha}S^{\alpha}_{s\sigma} \quad (\sigma = 1, 2, \cdots, r). \tag{4.6b}$$

We assume that the transformation group has a subgroup and $\varepsilon^\sigma(x) = \varepsilon^\rho_0\xi^\sigma_\rho(x)$, where ε^ρ_0 ($\rho = 1, 2, \cdots, r$) are numerical parameters of the Lie group, and $\xi^\sigma_\rho(x)$ are some functions. For example, the BRS transformation in Yang–Mills theories with a higher-order Lagrangian belongs to this category. Under this circumstance, the quantum strong conservation laws (4.6a) become

$$Q_\rho = \int \mathrm{d}^3x j_\sigma \xi^\sigma_\rho = \text{const} \quad (\rho = 1, 2, \cdots, r). \tag{4.7}$$

The quantum canonical equations of motion are not used when one deduces the expressions (4.7); hence the quantum strong conservation laws (4.7) hold true whether or not $\varphi^{\alpha}_{(s)}$ and $\pi^{(s)}_{\alpha}$ are solutions of the quantum canonical equations of this generalized constrained Hamiltonian system. Using the quantum canonical equations of motion, one has [9] $\delta I^{\rm P}_{\rm eff}/\delta\varphi^{\alpha}_{(s)} = 0$ and $\delta I^{\rm P}_{\rm eff}/\delta\pi^{(s)}_{\alpha} = 0$, and from the expressions (4.7), one can obtain the quantum (weak) conservation laws:

$$Q^w_\rho = \int \mathrm{d}^3x(\pi^{(s)}_{\alpha}S^{\alpha}_{s\sigma} - u^0_\sigma + \partial_\nu u^{0\nu}_\sigma - u^{0\nu}_\sigma\partial_\nu)\xi^\sigma_\rho = \text{const} \quad (\rho = 1, 2, \cdots, r). \tag{4.8}$$

Thus, if the effective canonical action is invariant under the just-mentioned transformation, then these quantum conservation laws coincide with the results derived by using the quantum canonical first Noether theorem for the global symmetry transformation in phase space [9]. We have

$$Q^w_\rho = \int \mathrm{d}^3x \pi^{(s)}_{\alpha}S^{\alpha}_{s\sigma}\xi^\sigma_\rho \quad (\rho = 1, 2, \cdots, r). \tag{4.9}$$

We have shown that in certain cases the quantum canonical NIs may be converted into quantum (weak) conservation laws even if the effective canonical action is not invariant under the specific transformation. This algorithm deriving quantum conservation laws differs from the canonical first Noether theorem at the quantum level [9].

For a system with a gauge-invariant higher-order Lagrangian, based on the quantum NIs (3.12) or (3.13) in configuration space, one can proceed in the same way as before to deduce the quantum conservation laws. Let us consider a local transformation in (3.2) with $\Delta x^\mu = 0$ and $E = 0$ and $V_\sigma = v_\sigma + v^\mu_\sigma\partial_u + v^{\mu\nu}_\sigma\partial_\mu\partial_\nu$ in (3.3); from (3.13) one can obtain the quantum conservation laws

$$\partial_\mu\left\{\left[\sum_{m=0}^{N-1}\prod_{\rm eff}^{\mu\nu(m)}\partial_{\nu(m)}S_\sigma + a^\mu_\sigma\frac{\delta I_{\rm eff}}{\delta\varphi} + \partial_\nu\left(a^{\mu\nu}_\sigma\frac{\delta I_{\rm eff}}{\delta\varphi}\right) - \left(a^{\mu\nu}_\sigma\frac{\delta I_{\rm eff}}{\delta\varphi}\right)\partial_\nu - v^\mu_\sigma + \partial_\nu v^{\mu\nu}_\sigma - v^{\mu\nu}_\sigma\partial_\nu\right]\varepsilon^\sigma(x)\right\} = 0. \tag{4.10}$$

This holds if $\varepsilon^\sigma(x) = \varepsilon^\rho_0\xi^\sigma_\rho$, where the ε^ρ_0 are parameters. Using the quantum equations of motion [9], $\delta I_{\rm eff}/\delta\varphi = 0$, from (4.10) one can obtain the quantum (weak) conservation laws in configuration space.

5 Yang–Mills fields with higher-order derivatives

The Lagrangian of Yang–Mills fields with a higher-order Lagrangian is given by [17]

$$\mathcal{L} = -\frac{1}{4\kappa^2}D^a_{b\mu}F^b_{\nu\lambda}D^{a\mu}_c F^{c\lambda\nu}, \tag{5.1}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc}A^b_\mu A^c_\nu, \tag{5.2}$$

$$D^a_{b\mu} = \delta^a_b\partial_\mu + f^a_{cb}A^c_\mu, \tag{5.3}$$

the phase-space generating functional of the Green functions for this system can be written as [9]

$$Z[J, \xi, \bar{\xi}, \eta] = \int \mathcal{D}A^a_\mu\mathcal{D}A^a_{(1)\mu}\mathcal{D}\pi^\mu_a\mathcal{D}\pi^{(1)\mu}_a\mathcal{D}C^a\mathcal{D}\bar{C}^a\mathcal{D}\lambda^m \times \exp\left\{\mathrm{i}\int \mathrm{d}^4x(\mathcal{L}^{\rm P}_{\rm eff} + J^\mu_a A^a_\mu + \bar{C}^a\xi_a + \bar{\xi}_a C^a + \eta^m\lambda_m)\right\}, \tag{5.4}$$

where

$$\mathcal{L}^{\rm P}_{\rm eff} = \mathcal{L}^{\rm P} + \mathcal{L}_f + \mathcal{L}_m + \mathcal{L}_{\rm gh}, \tag{5.5}$$

$$\mathcal{L}^{\rm P} = \pi^\mu_a\dot{A}^a_\mu + \pi^{(1)\mu}_a\dot{A}_{(1)\mu} - \mathcal{H}_{\rm c}, \tag{5.6}$$

$$\mathcal{L}_f = -\frac{1}{2\alpha_2}(\partial^\mu A^\alpha_\mu)^2, \tag{5.7}$$

$$\mathcal{L}_m = \lambda^a_1\Phi^{(1)}_{a_1} + \lambda^a_2\Phi^{(2)}_{a_2} - \frac{1}{2\alpha_1}(\Phi^G_{a_1})^2, \tag{5.8}$$

$$\mathcal{L}_{\rm gh} = -\partial^\mu\bar{C}_a D^a_{b\mu}C_b, \tag{5.9}$$

and $\mathcal{H}_c$ is the canonical Hamiltonian density for the Lagrangian (5.1), π_a^μ and $\pi_{(1)a}^\mu$ are the canonical momenta conjugate to A_μ^a and $A_{(1)\mu}^a = \dot{A}_\mu^a$, respectively, and $\{\Phi\}$ and $\{\Phi^G\}$ are constraints and gauge conditions, respectively [17]; the $\{\lambda\}$ are multiplier fields, and finally C^a and $\bar{C}^a$ are ghost fields. In the expression (5.4) we have introduced exterior sources only for the fields.

It is easy to check that $\mathcal{L}^{\mathrm{P}}$ and $\mathcal{L}_{\mathrm{gh}}$ are invariant under the following transformation [9]:

$$A_\mu^{a'}(x) = A_\mu^a(x) + D_{\sigma\mu}^a \varepsilon^\sigma(x), \tag{5.10a}$$

$$A_{(1)\mu}^{a'}(x) = A_{(1)\mu}^a(x) + \partial_0 D_{\sigma\mu}^a \varepsilon^\sigma(x), \tag{5.10b}$$

$$\pi_a^{\mu'}(x) = \pi_a^\mu(x) + f_{\sigma c}^a \pi_c^\mu(x)\varepsilon^\sigma(x) + f_{\sigma c}^a \pi_c^{(1)\mu}(x)\dot{\varepsilon}^\sigma(x), \tag{5.10c}$$

$$\pi_a^{(1)\mu'}(x) = \pi_a^{(1)\mu}(x) + f_{\sigma c}^a \pi_c^{(1)\mu}(x)\varepsilon^\sigma(x), \tag{5.10d}$$

$$C^{a'}(x) = C^a(x) + \mathrm{i}(T_\sigma)_b^a C^b(x)\varepsilon^\sigma(x), \tag{5.10e}$$

$$\bar{C}^{a'}(x) = \bar{C}^a(x) - \mathrm{i}\bar{C}^b(x)(T_\sigma)_b^a\varepsilon^\sigma(x) + \mathrm{i}\int \mathrm{d}^4y \varDelta_0(x,y)\partial_\mu[\bar{C}^b(x)(T_\sigma)_b^a\partial^\mu\varepsilon^\sigma(x)], \tag{5.10f}$$

where T_σ are the representation matrices of the generators of the gauge group, and f_{bc}^a are the structure constants of the gauge group. Let us put $\varepsilon^\sigma(x) = \varepsilon_0 C^\sigma(x)$, where ε_0 is a numerical parameter; then the transformation (5.10) will be converted into a global one. As is well known, the variations of first-class constraints under the gauge transformation (5.10a)–(5.10d) are within the constraint hypersurface [24]; thus $\delta\mathcal{L}_m \approx 0$ and $\delta\mathcal{L}_f \approx 0$ under the transformation (5.10). Therefore, one has $\delta I_{\mathrm{eff}}^{\mathrm{P}} \approx 0$ under the transformation (5.10). Using the quantum canonical equations of the system, from (4.9) one obtains the quantum conserved quantity

$$Q = \int \mathrm{d}^4x \Bigg\{ \pi_a^\mu D_{\sigma\mu}^a C^\sigma + \pi_a^{(1)\mu}\partial_0(D_{\sigma\mu}^a C^\sigma) + \mathrm{i}\pi_a(T_\sigma)_b^a C^b C^\sigma - \mathrm{i}\bar{\pi}_a \Bigg[\bar{C}^b(T_\sigma)_b^a C^\sigma - \int \mathrm{d}^4y \varDelta_0(x,y)\partial_\mu(\bar{C}^b(x)(T_\sigma)_b^a\partial^\mu\bar{C}^\sigma(x)) \Bigg] \Bigg\}, \tag{5.11}$$

where $\pi_a^\mu, \pi_a^{(1)\mu}, \pi_a$ and $\bar{\pi}_a$ are canonical momenta conjugate to the fields $A_\mu^a, \dot{A}_\mu^a, C^a$ and $\bar{C}^a$, respectively, and

$$\pi_a^0 = \frac{1}{\kappa^2} D_{aj}^b D_{b0}^c F_c^{j0}, \tag{5.12}$$

$$\pi_a^j = \frac{1}{\kappa^2}(D_a^{bj} D_{bj}^e F_e^{0i} + D_{bj}^a D_{b0}^e F_e^{ij}) - D_{a0}^b \pi_b^{(1)i} + F_a^{0i}, \tag{5.13}$$

$$\pi_a^{(1)0} = 0, \tag{5.14}$$

$$\pi_a^{(1)i} = \frac{1}{\kappa^2} D_{bj}^a F_b^{ij}, \tag{5.15}$$

$$\pi_a = -\dot{\bar{C}}^a, \tag{5.16}$$

$$\bar{\pi}_a = D_{b0}^a C^b. \tag{5.17}$$

This quantum conserved quantity (5.11) can also be derived by using the quantum canonical first Noether theorem for the global transformation [8,9]. Here we provide another algorithm to derive these quantum conservation laws.

6 Non-Abelian Chern–Simons theory with higher-order derivatives

Numerous recent work on $(2+1)$-dimensional gauge theories with CS terms in the Lagrangian has revealed the occurrence of fractional spin and statistics [25]. These theories are frequently used in condensed matter studies, such as the quantum Hall effect and high-T_c superconductivity. Now we give some preliminary application of quantum NIs to non-Abelian CS theory with higher-order derivatives.

Let us consider the CS gauge fields A_μ^a coupled to the scalar field φ whose higher-order Lagrangian is given by [9]

$$\mathcal{L} = -\frac{c^2}{4\pi} D_\rho F_{\mu\nu}^a D^\rho F^{a\mu\nu} - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\rho}\left(\partial_\mu A_\nu^a A_\rho^a + \frac{1}{3} f_{bc}^a A_\mu^a A_\nu^b A_\rho^c\right) + (D_\mu\varphi)^+(D^\mu\varphi), \tag{6.1}$$

where

$$D_\mu\varphi^\alpha = \partial^\mu\varphi^\alpha + A_\mu^\gamma T_{\alpha\beta}^\gamma \varphi^\beta, \tag{6.2}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c, \tag{6.3}$$

and the T^γ are the generators of the gauge group; the f_{bc}^a are the structure constants of the gauge group. The gauge invariance of the non-Abelian CS term requires the quantization of the dimensionless constant k, $k = \frac{n}{4\pi}$ $(n \in Z)$ [26].

The phase-space generating functional of the Green function for this system can be written as [9]

$$Z[J] = \int \mathcal{D}A_\mu^a \mathcal{D}P^{a\mu}\mathcal{D}B_\mu^a\mathcal{D}Q^{a\mu}\mathcal{D}\varphi\mathcal{D}\pi^+\mathcal{D}\varphi^+\mathcal{D}\pi\mathcal{D}\lambda\mathcal{D}\bar{C}\mathcal{D}C \times \exp\left\{\mathrm{i}\int \mathrm{d}^3x(\mathcal{L}_{\mathrm{eff}}^{\mathrm{P}} + J_{1a}^\mu A_\mu^a + J_{2a}^\mu B_\mu^a + J_1^+\varphi + \varphi^+ J_1 + \bar{J}_{3a}C^a + \bar{C}^a J_{3a})\right\}, \tag{6.4}$$

where $P^{a\mu}, Q^{a\mu}, \pi^+$ and π are the canonical momenta conjugate to $A_\mu^a, B_\mu^a = \dot{A}_\mu^a, \varphi$ and φ^+, respectively, the $\{\lambda\}$ are the multiplier fields, $\bar{C}^a$ and C^a are ghost fields, and

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{P}} = \mathcal{L}^{\mathrm{P}} + \mathcal{L}_g + \mathcal{L}_m + \mathcal{L}_{\mathrm{gh}}, \tag{6.5}$$

$$\mathcal{L}^{\mathrm{P}} = B_\mu^a P^{a\mu} + \dot{B}_\mu^a Q^{a\mu} + \dot{\varphi}^+\pi + \pi^+\dot{\varphi} - \mathcal{H}_c, \tag{6.6}$$

$$\begin{aligned}\mathcal{L}_g &= -\frac{1}{2\alpha_2}(\partial^\mu A^a_\mu)^2,\\ \mathcal{L}_m &= \lambda^a_0 \Lambda^{(0)a} + \lambda^{(1)a}_1 \Lambda^{(1)a} + \lambda^a_2 \Lambda^{(2)a}\\ &\quad -\frac{1}{2\alpha_0}(\Omega^a_0)^2 - \frac{1}{2\alpha_1}(\Omega^a_1)^2,\\ \mathcal{L}_{\rm gh} &= -\partial^\mu \bar{C}^a D^a_{b\mu} C^b,\end{aligned} \tag{6.7}$$

and $\mathcal{H}_{\rm c}$ is a generalized canonical Hamiltonian density; $\Lambda^{(i)a} \approx 0$ $(i = 0, 1, 2)$ are first-class constraints in phase space, while $\Omega^a_i \approx 0$ are gauge conditions [9].

Let us consider the BRS (Becchi–Rouet–Stora) transformation

$$\delta A^a_\mu = -\tau D^a_{b\mu} C^b, \quad \delta B^a_\mu = -\tau \partial_0 (D^a_{b\mu} C^b), \tag{6.8a}$$

$$\delta\varphi = -\mathrm{i}\tau T^a C^a \varphi, \quad \delta\varphi^+ = \mathrm{i}\tau\varphi^+ T^a C^a, \tag{6.8b}$$

$$\delta C^a = \frac{1}{2} f^a_{bc} C^b C^c, \quad \delta\bar{C}^a = \frac{1}{\alpha_2}\partial^\mu A^a_\mu, \tag{6.8c}$$

where τ is a Grassmann parameter, and the corresponding transformation of the canonical momenta is also taken into account. Under the BRS transformation the action connected with the sum of the terms $\mathcal{L}^{\rm P}, \mathcal{L}_g$ and $\mathcal{L}_{\rm gh}$ is invariant at the quantum level. The variations of the first-class constraints under the transformation (6.8_a) are within the constraint hypersurface; thus $\delta\mathcal{L}_m \approx 0$ under the transformation (6.8), and hence we have $\delta I^{\rm P}_{\rm eff} \approx 0$ under the transformation (6.8). From (4.9) one gets the conserved BRS quantity on the constraint hypersurface at the quantum level

$$\begin{aligned}Q_{\rm B} = \int \mathrm{d}^2 x (&P^\mu_a \delta A^a_\mu + Q^\mu_a \delta B^a_\mu + \pi^+ \delta\varphi + \delta\varphi^+ \pi\\ &+ \bar{R}_a \delta C^a + \delta \bar{C}^a R_a),\end{aligned} \tag{6.9}$$

where $\bar{R}_a$ and R_a are canonical momenta conjugate to C^a and $\bar{C}^a$, respectively. This result can also be derived by using the quantum canonical first Noether theorem [9].

If we only consider the transformation of A^a_μ, φ and φ^+ in the BRS transformation, fixing the ghost fields,

$$\begin{aligned}&\delta A^a_\mu = -\tau D^a_{b\mu} C^b, \quad \delta B^a_\mu = -\tau\partial_0 (D^a_{b\mu} C^b),\\ &\delta\varphi = -\mathrm{i}\tau T^a C^a \varphi, \quad \delta\varphi^+ = \mathrm{i}\tau \varphi^+ T^a C^a,\\ &\delta C^a = \delta\bar{C}^a = 0.\end{aligned} \tag{6.10}$$

Under the transformation (6.10), the change of $\mathcal{L}^{\rm P}_{\rm eff}$ in the theory is given by

$$\delta\mathcal{L}^{\rm P}_{\rm eff} = U_a \varepsilon^a(x) = W_a \varepsilon^a(x) + f^a_{bc}\partial^\mu \bar{C}^a C^b \partial_\mu \varepsilon^c(x), \tag{6.11}$$

within the constraint hypersurface, where $\varepsilon^a(x) = \tau C^a(x)$, and the W_a do not contain the derivatives of the $\varepsilon^a(x)$, from (4.8) one obtains the quantum conserved PBRS (P stands for "partial") quantity

$$\begin{aligned}Q = \int \mathrm{d}^2 x \Big(&P^\mu_a \delta A^a_\mu + Q^\mu_a \delta B^a_\mu + \pi^+\delta\varphi + \delta\varphi^+\pi\\ &- f^a_{bc} \dot{\bar{C}}^a C^b C^c\Big).\end{aligned} \tag{6.12}$$

This quantum conserved quantity Q differs from $Q_{\rm B}$ in (6.9).

The above quantum conserved quantities $Q_{\rm B}$ and Q can also be deduced by using the configuration-space generating functional as discussed in Sect. 4. Based on the classical Noether identities, the conserved PBRS charge has been discussed in semi-classical theories [5,6]. The above results are valid at the quantum level.

As is well known, the BRS charge annihilates the vacuum state; this conserved PBRS charge may also impose some supplementary condition on the physical state as well as the BRS charge and the ghost charge. The study along this line is in progress.

The effective Lagrangian for the model (6.1) in configuration space can be obtained by using the FP trick in Lorentz gauge through a transformation of the path integral:

$$\mathcal{L}_{\rm eff} = \mathcal{L} - \partial^\mu \bar{C}^a D^a_{b\mu} C^b - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2. \tag{6.13}$$

It is easy to check that the action connected with the sum of the first two terms on the right hand side of (6.13) in the theory is invariant under the following transformation at the quantum level [24]:

$$\delta A^a_\mu = D^a_{\sigma\mu}\varepsilon^\sigma(x), \tag{6.14a}$$

$$\delta\varphi = -\mathrm{i}T^\sigma \varphi\varepsilon^\sigma(x), \quad \delta\varphi^+ = \mathrm{i}\varphi^+ T^\sigma \varepsilon^\sigma(x), \tag{6.14b}$$

$$\delta C^a = \mathrm{i}C^b (T_\sigma)^a_b \varepsilon^\sigma(x), \tag{6.14c}$$

$$\begin{aligned}\delta\bar{C}^a &= \mathrm{i}\bar{C}^b (T_\sigma)^a_b \varepsilon^\sigma(x)\\ &\quad - \mathrm{i}\int \mathrm{d}^3 y \Delta_0(x,y)\partial_\mu[\bar{C}^b(y)(T_\sigma)^a_b \partial^\mu \varepsilon^\sigma(y)].\end{aligned} \tag{6.14d}$$

From the quantum NIs in configuration space (3.12), one has

$$\begin{aligned}&-\mathrm{i}\frac{\delta I_{\rm eff}}{\delta\varphi(x)} T^\sigma \varphi(x) + \mathrm{i}\varphi^+(x) T^\sigma \frac{\delta I_{\rm eff}}{\delta\varphi^+(x)} + \tilde{D}^a_{\sigma\mu}\left(\frac{\delta I_{\rm eff}}{\delta A^a_\mu(x)}\right)\\ &+ \mathrm{i}\frac{\delta I_{\rm eff}}{\delta C^a(x)}(T_\sigma)^a_b C^b(x) - \mathrm{i}\bar{C}^b(x)(T_\sigma)^a_b \frac{\delta I_{\rm eff}}{\delta\bar{C}^a(x)}\\ &+ \int \mathrm{d}^3 y N^a_b \left[\partial_\mu\left(\frac{\partial\mathcal{L}_{\rm eff}}{\partial\bar{C}^a_{,\mu}}\right)\Delta_0(y,x)\right] = \frac{1}{\alpha}\tilde{D}^a_{\sigma\mu}\partial^\mu(\partial^\nu A^a_\nu),\end{aligned} \tag{6.15}$$

where

$$\begin{aligned}N^a_\sigma &= \partial_\mu[\bar{C}^b (T_\sigma)^a_b \partial^\mu],\\ \tilde{D}^a_{\sigma\mu} &= -\delta^a_\sigma \partial_\mu + f^a_{\sigma c} A^c_\mu.\end{aligned} \tag{6.16}$$

Using the quantum equations of motion, from (6.15) one obtains the quantum conserved quantity in the Lorentz gauge

$$Q'_\sigma = \int \mathrm{d}^2 x \int \mathrm{d}^3 y \bar{C}^b (T_\sigma)^a_b \partial_{x_0}\left[\partial_{y_\nu}\left(\frac{\partial\mathcal{L}_{\rm eff}}{\partial\bar{C}^a_{,y_\nu}}\right)\Delta_0(y,x)\right]. \tag{6.17}$$

Substituting (6.13) into (6.17), one gets

$$Q'_\sigma = \int \mathrm{d}^2 x \int \mathrm{d}^3 y \bar{C}^b(x)(T_\sigma)^a_b (\partial^\nu D^a_{e\nu} C^e)\partial_{x_0}\Delta(y,x). \tag{6.18}$$

7 Conclusions and discussion

In this paper we have studied the transformation properties under a local and non-local transformation at the quantum level for a system with a regular/singular higher-order Lagrangian. The path integrals provide a useful tool, where the main ingredient is the classical action together with the measure in the space of field configurations. The phase-space path integrals are more fundamental than configuration-space path integrals. In certain integrable cases, the phase-space path integral can be simplified by carrying out explicit integration over canonical momenta, and then the phase-space path integral can be represented in the form of a path integral only over the field variables of the expression containing a certain Lagrangian (or effective Lagrangian) in configuration space. In the more general case, especially for a system with a singular higher-order Lagrangian with complicated constraints, it is very difficult or even impossible to carry out the integration over the canonical momenta. However, for a gauge-invariant system one can conveniently use the FP trick to formulate its path-integral quantization in configuration space. Based on the phase-space generating functional of Green function obtained by using the FS path-integral quantization method for a system with a singular higher-order Lagrangian, the quantum canonical NIs under the local and non-local transformation in phase space for a system with a regular/singular higher-order Lagrangian are derived, respectively. Based on the configuration-space generating functional of the Green function obtained by using the FP trick for a gauge-invariant system with a higher-order Lagrangian, the quantum NIs under the local and non-local transformation in configuration space for such a system are deduced. For a system with a singular higher-order Lagrangian one must use an effective action instead of the classical action in quantum NIs. The results hold true regardless as to whether the Jacobian of the transformation is equal to unity or not. It is pointed out that in certain cases, according to the quantum NIs, using the quantum equations of motion one can obtain the quantum conservation laws. This algorithm to deduce the conservation laws at the quantum level differs from the quantum first Noether theorem. The applications of the theory to Yang–Mills fields and non-Abelian CS gauge fields coupled to a scalar field with higher-order derivatives have been presented, and some quantum conserved quantities for a local and non-local transformation are obtained.

In the non-Abelian CS theory, the quantum conserved angular momentum arising from the invariance of a spatial rotation can also be deduced. This conserved angular momentum at the quantum level differs from the classical Noether one in that one needs to take into account the contribution of the angular momentum of ghost fields in a system with a higher-order non-Abelian CS term coupled to matter fields. Recent work [27] has studied the occurrence of fractional spin for non-Abelian CS theories in classical theories. Whether the fractional spin properties for non-Abelian CS theories are always valid at the quantum level is a question which needs further study.

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